POISSON STATISTICS OF EIGENVALUES IN THE HIERARCHICAL DYSON MODEL

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ABSTRACT. Let (X, d) be a locally compact separable ultrametric space. Given a measure m on X and a function C defined on the set $\mathcal{B}$ of all balls $B \subset X$ we consider the hierarchical Laplacian $L = L_C$. The operator L acts in $L^2(X, m)$, is essentially self-adjoint, and has a purely point spectrum. Choosing a family $\{\varepsilon(B)\}_{B \in \mathcal{B}}$ of i.i.d. random variables, we define the perturbed function $\mathcal{C}(B) = C(B)(1 + \varepsilon(B))$ and the perturbed hierarchical Laplacian $\mathcal{L} = L_{\mathcal{C}}$. All outcomes of the perturbed operator $\mathcal{L}$ are hierarchical Laplacians. In particular they all have purely point spectrum. We study the empirical point process M defined in terms of $\mathcal{L}$-eigenvalues. Under some natural assumptions M can be approximated by a Poisson point process. Using a result of Arratia, Goldstein, and Gordon based on the Chen-Stein method, we provide total variation convergence rates for the Poisson approximation. We apply our theory to random perturbations of the operator $\mathfrak{D}^\alpha$, the p-adic fractional derivative of order $\alpha > 0$. This operator, related to the concept of p-adic Quantum Mechanics, is a hierarchical Laplacian which acts in $L^2(X, m)$ where $X = \mathbb{Q}_p$ is the field of p-adic numbers and m is Haar measure. It is translation invariant and the set $\mathsf{Spec}(\mathfrak{D}^\alpha)$ consists of eigenvalues $p^{\alpha k}$, $k \in \mathbb{Z}$, each of which has infinite multiplicity.

CONTENTS

1. Introduction

The concept of hierarchical lattice and hierarchical distance was proposed by F.J. Dyson in his famous papers on the phase transition for a one-dimensional ferromagnetic model with long range interaction [17, 18]. The notion of the hierarchical Laplacian L, which is closely related to Dyson's model, was studied in several mathematical papers [11, 23, 24, 25, 31, 3, 32, 22]. These papers contain some basic information about L (the spectrum, Markov semigroup, resolvent, etc...) in the case when the state space (X, d, m) is discrete and satisfies some symmetry conditions such as homogeneity and self-similarity. As was noticed in [14, 15], these symmetry conditions imply that (X, d, m) can be identified with some discrete infinitely generated Abelian group G equipped with a translation invariant ultrametric and measure. The Markov semigroup $\{P^t = \exp(-tL)\}_{t \geq 0}$ acting on $L^2(G, m)$ becomes symmetric, translation invariant, and isotropic. In particular, $\mathsf{Spec}(L)$ is pure point and all eigenvalues have infinite multiplicity.

The main goal of the papers mentioned above was to study the corresponding Anderson Hamiltonian $H = L + V$; the hierarchical Laplacian L plus random potential V. There was a hope to detect for such operators the spectral bifurcation from the pure point spectrum to the continuous one, i.e. to justify the famous Anderson conjecture. Unfortunately, the opposite result was true. Under mild technical conditions, the hierarchical Anderson Hamiltonian has a pure point spectrum—the phenomenon of localization (see [32, 26, 27, 24]). Moreover, the local statistics of the spectrum of H are Poissonian [25], which is always deemed a manifestation of the spectral localization [2, 30].

We study a class of operators introduced in [7], the random hierarchical Laplacians, which demonstrate several new spectral effects. The spectra of such operators is still pure point (with compactly supported eigenfunctions), but in contrast to the deterministic case there exists a continuous density of states. This density detects the spectral bifurcation from the pure point spectrum to the continuous one. The eigenvalues *form locally* a Poisson point process with intensity given by the density of states. That is, the empirical point process defined in terms of the eigenvalues is approximated by a Poisson point process. In this paper we provide an error bound for the Poisson approximation in terms of the total variation distance; see Theorem 4.1. We prove Theorem 4.1 by applying a result due to R. Arratia, L. Goldstein and L. Gordon in [4], which studies Poisson approximations using the Chen-Stein method.

Throughout the paper we assume that (X, d) is a locally compact, non-compact, and separable ultrametric space. Recall that a metric d is called an *ultrametric* if it satisfies the ultrametric inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}, \tag{1.1}$$

which is stronger than the usual triangle inequality. We also assume that the ultrametric d is *proper*, i.e. each closed d-ball is a compact set.

Let m be a Radon measure on (X, d) such that

- $m(B) > 0$ for each ball B of positive diameter.
- $m(\{x\}) = 0$ if and only if x is not an isolated point.

Let $\mathcal{B}$ be the set of all balls having positive measure. Our assumptions imply that the set $\mathcal{B}$ is countable. Let $C : \mathcal{B} \to (0, \infty)$ be a function which satisfies the following assumptions:

$$\lambda(B) := \sum_{T \in \mathcal{B}:\ B \subseteq T} C(T) < \infty$$

and

$$\sup\{\lambda(B) : B \in \mathcal{B} \text{ and } x \in B\} = \infty$$

holds for all $B \in \mathcal{B}$ and non-isolated $x \in X$. As was shown in [10], the set of functions C which satisfy the above assumptions is quite rich.

Let $\mathcal{D}$ be the set of all locally constant functions having compact support. Given the data (X, m, C) one defines (pointwise) the hierarchical Laplacian

$$L_C f(x) := \sum_{B \in \mathcal{B}:\ x \in B} C(B) \left(f(x) - \frac{1}{m(B)} \int_B f \, dm \right), \quad f \in \mathcal{D}. \qquad (1.2)$$

The operator $(L_C, \mathcal{D})$ acts in $L^2 = L^2(X, m)$, is symmetric, and admits a complete system of eigenfunctions $\{f_{B,B'}\}$, defined as

$$f_{B,B'} = \frac{1}{m(B)} \mathbf{1}_B - \frac{1}{m(B')} \mathbf{1}_{B'}, \qquad (1.3)$$

where $B \subset B'$ run over all nearest neighboring balls in $\mathcal{B}$. The family $\{f_{B,B'}\}$ is called the Haar system associated to (X, d, m). The eigenvalue $\lambda(B')$ corresponding to the Haar function $f_{B,B'}$ is computed as

$$\lambda(B') = \sum_{T \in \mathcal{B}:\ B' \subseteq T} C(T). \qquad (1.4)$$

Since the Haar system $\{f_{B,B'}\}$ is complete, we conclude that $(L_C, \mathcal{D})$ is an essentially self-adjoint operator in L^2. By a slight abuse of notation, we write $(L_C, \mathsf{Dom}_{L_C})$ for its unique self-adjoint extension. For a more detailed discussion we refer the reader to [10]. See also the related papers [8] and [9].

Observe that to define the functions $C(B)$, $\lambda(B)$, and therefore the operator $(L_C, \mathsf{Dom}_{L_C})$, we do not need to specify the ultrametric d. What is needed is the family of balls $\mathcal{B}$, which can be identical for two different ultrametrics d and d'. In particular, given the data $(X, \mathcal{B}, m)$, and choosing the function

$$C(B) = \frac{1}{m(B)} - \frac{1}{m(B')},$$

where $B \subset B'$ are any two nearest neighbor balls in $\mathcal{B}$, we obtain the hierarchical Laplacian $(L_C, \mathsf{Dom}_{L_C})$ satisfying

$$\lambda(B) = \frac{1}{m(B)}.$$

We refer to $(L_C, \mathsf{Dom}_{L_C})$ as the standard hierarchical Laplacian associated with the data $(X, \mathcal{B}, m)$.

Notation.

Denoting the positive integers by $\mathbb{N}$, we write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N}_{\geq 2} = \mathbb{N} \setminus \{1\}$. For a function $f : \mathbb{R} \to \mathbb{R}$, define $\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$. For a real-valued random variable X, we let $\mathscr{L}(X)$ denote the law of that random variable. The law of a Poisson random variable with mean λ is denoted $\mathrm{Poi}(\lambda)$. The *total variation distance* between probability measures μ and ν on $\mathbb{R}$ with the Borel σ-field $\mathscr{B}(\mathbb{R})$ is defined as

$$\|\mu - \nu\|_{TV} = \sup_{A \in \mathscr{B}(\mathbb{R})} |\mu(A) - \nu(A)| .$$

When μ and ν have countable support S, total variation is half the L^1 distance between the associated mass functions:

$$\|\mu - \nu\|_{TV} = \frac{1}{2} \sum_{x \in S} |\mu(\{x\}) - \nu(\{x\})| .$$

Outline.

In Section 2 we specify some spectral properties of the hierarchical Laplacian L_C assuming that the ultrametric measure space where it acts and the Laplacian itself both satisfy certain symmetry conditions (homogeneity and self-similarity). As an example, we consider the operator $\mathfrak{D}^\alpha$ of the p-adic fractional derivative of order $\alpha > 0$. This operator is related to the concept of p-adic Quantum Mechanics and was introduced by V.S. Vladimirov; see [39], [40] and [41]. $\mathfrak{D}^\alpha$ is a homogeneous hierarchical Laplacian which acts in $L^2(X, m)$ where $X = \mathbb{Q}_p$ is the field of p-adic numbers and m is its Haar measure. It is translation invariant and the set $\mathsf{Spec}(\mathfrak{D}^\alpha)$ consists of the eigenvalues $\{p^{\alpha k}\}_{k \in \mathbb{Z}}$, each having infinite multiplicity, and contains 0 as an accumulation point.

In Section 3, given a homogeneous Laplacian L_C and a family $\{\varepsilon(B)\}_{B \in \mathcal{B}}$ of symmetric i.i.d. random variables, we define a perturbed function $\mathcal{C}(B) = C(B)(1 + \varepsilon(B))$ and a perturbed Laplacian $\mathcal{L} = L_\mathcal{C}$. For each ω, the operator $\mathcal{L}(\omega) = L_{\mathcal{C}(\omega)}$ is a hierarchical Laplacian, whence it has a pure point spectrum. On the other hand, for some ω it may fail to be homogeneous. In particular, the set of its eigenvalues may form dense subsets in certain intervals [10]. Following [7] we recall some basic notions and properties associated to the perturbed operator $\mathcal{L}$.

Our main result appears in Section 4. We fix a horocycle H (the set of all balls having the same diameter) and define the empirical process

$$M_O = \sum_{B \subseteq O : B \in H} \delta_{\lambda(B)}, \ O \in \mathcal{B},$$

associated with eigenvalues $\lambda(B)$ of the perturbed operator $\mathcal{L}$. Here δ_a denotes the point mass at a, so that for any interval I, $\delta_{\lambda(B)}(I) = 1\{\lambda(B) \in I\}$ is the indicator that $\lambda(B)$ belongs to I and $M_O(I)$ is the number of $\lambda(B)$, $B \subseteq O$, which fall in I. Under mild assumptions, it was shown in [7] that for an appropriate sequence of intervals I_O, when $\mathbb{E}[M_O(I_O)]$ converges to some value $\lambda > 0$ as $O \to \varpi$, the random variable $M_O(I_O)$ converges in law to a Poisson random variable $\mathrm{Poi}(\lambda)$. In Theorem 4.1, we give an upper bound on the total variation distance

$$\|\mathscr{L}(M_O(I_O)) - \mathrm{Poi}(\lambda)\|_{TV} .$$

Random perturbations of the operator $\mathfrak{D}^\alpha$ of the p-adic fractional derivative of order $\alpha > 1$ from Section 2 provide an example where our result can be applied.

Section 5 provides a proof of Theorem 4.1. The proof is based on a result by R. Arratia, L. Goldstein, and L. Gordon [4], who study Poisson approximation of random variables using the Chen-Stein method.

2. Homogeneous Laplacian

In this section we specify some spectral properties of the hierarchical Laplacian L_C assuming that the (non-compact) ultrametric measure space (X, d, m) where it acts and the Laplacian by itself satisfy the following symmetry conditions:

- The group of isometries of (X, d) acts transitively on X.
- Both the reference measure m and the function $C(B)$ are invariant under the action of isometries.

When the ultrametric measure space (X, d, m) and hierarchical Laplacian L_C satisfy both of these conditions, we call them *homogeneous*. The first assumption implies that (X, d) is either discrete or perfect. Basic examples that we have in mind are

- $X = \mathbb{Z}_p$ – the compact group of p-adic integers.
- $X = \mathbb{Q}_p$ – the ring of p-adic numbers.[1]
- $X = \mathbb{Q}_p/\mathbb{Z}_p \cong \mathbb{Z}(p^\infty)$ – the multiplicative group of p^nth roots of unity, $n \in \mathbb{N}$, considered in the discrete topology.
- $X = S_\infty$ – the infinite symmetric group.

As was noticed in [14] and [15], our assumptions imply that the measure space (X, m) can be identified with a locally compact totally disconnected Abelian group G equipped with its Haar measure. Notice that the group G is not unique. As a possible choice of G when, for instance, X is *perfect*, one can take the Abelian group

$$G = \varinjlim_{\ell \to -\infty} \left(\prod_{k \geq \ell} \mathbb{Z}(n_k) \right), \tag{2.1}$$

where $\mathbb{Z}(n_k)$ are (nontrivial) cyclic groups and $\{n_k\}_{k \in \mathbb{Z}}$ is an appropriately chosen double sided sequence of integers. The canonical ultrametric structure on G is defined by the descending sequence of compact subgroups

$$G_\ell = \prod_{k \geq \ell} \mathbb{Z}(n_k).$$

Namely, the groups G_ℓ and their cosets $\{a + G_\ell\}$ form the collection $\mathcal{B}$ of all clopen balls.

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[1]Though we sometimes refer to the field $\mathbb{Q}_p$, the assumption that p is prime is never needed in this paper.

There is a natural ultrametric structure associated to the chain of (small) subgroups G_ℓ of G. One defines the *absolute value* $|a|$ for the elements a of G,

$$|a| = \begin{cases} 0 & \text{if} & a = 0, \\ m(G_\ell) & \text{if} & a \in G_\ell \setminus G_{\ell+1}. \end{cases}$$

The absolute value $|a|$ satisfies the ultrametric inequality

$$|a + b| \leq \max\{|a|, |b|\}.$$

It is also clear that $|a| = |-a|$ and that $d(a, b) := |a - b|$ is an ultrametric that gives (G, m) the structure of a homogeneous ultrametric measure space as defined above. In particular, for any ball B we have

$$m(B) = \mathsf{diam}(B).$$

Choosing the Haar measure m such that $m(G_0) = 1$, we compute $m(G_\ell)$ for any $\ell \neq 0$ as follows:

$$m(G_\ell) = \begin{cases} n_\ell...n_{-1} & \text{if} & \ell < 0, \\ (n_{\ell-1}...n_0)^{-1} & \text{if} & \ell > 0. \end{cases}$$

Recall that in the classical setting $X = \mathbb{Q}_p$, we chose $G = \bigcup_{\ell \in \mathbb{Z}} G_\ell$, $G_\ell = p^\ell \mathbb{Z}_p$, and set

$$|a| = p^{-n(a)}, \text{ where } n(a) = \max\{\ell : a \in G_\ell\}.$$

In this case, the quantity $|a|$ becomes a pseudonorm: $|ab| \leq |a|\,|b|$. It is a norm ($|ab| = |a|\,|b|$) if p is a prime number—the basic property in p-adic analysis and its applications.

We recall that to any given ultrametric space (X, d) one associates in a standard fashion a tree $\mathcal{T}(X)$ (see Figure 1). The vertices of the tree are metric balls, and thus in our case they are the cosets $\{a + G_\ell : a \in G, \ell \in \mathbb{Z}\}$. The ascending sequence of subgroups $\{G_\ell : l = 0, -1, -2, ...\}$ identifies a special boundary point, which we denote ϖ. With respect to this special point we consider the horocycles of the tree. A *horocycle* in this case is the set of vertices consisting of the balls of a given diameter; in other words, the cosets relative to the same subgroup G_ℓ. Thus, for fixed ℓ, the horocycle is $H_\ell = \{a + G_\ell : a \in G\}$. The boundary $\partial \mathcal{T}(G)$ can be identified with the one-point compactification $G \cup \{\varpi\}$ of G. We refer to [14], [15], and [10] for a complete treatment of the association between an ultrametric space and the tree of its metric balls. The most complete source for the basic definitions related to the geometry of trees is [12]; see also [42].

Let L_C be a homogeneous hierarchical Laplacian. Thanks to the homogeneity property, $C(A) = C(B)$ for any two balls which belong to the same horocycle H. The same of course is true for the eigenvalues $\lambda(A)$ and $\lambda(B)$. We set $C_H = C(B)$ and $\lambda_H = \lambda(B)$ for any ball $B \in H$. When $H = H_k$ we will also write $c_k = C_{H_k}$ and $\lambda_k = \lambda_{H_k}$. In this notation

$$\lambda_k = \sum_{\ell \leq k} c_\ell.$$

Each ball $B \in H_k$ has n_k nearest neighbors $B_i \subset B$. The system of eigenfunctions $\{f_{B_i,B} : i = 1, ..., n_k\}$ corresponding to $\lambda(B)$ is not orthogonal: Its linear span $\mathcal{H}(B) \subset L^2$ has dimension $n_k - 1$. For two different balls S and T, the eigenspaces $\mathcal{H}(S)$ and
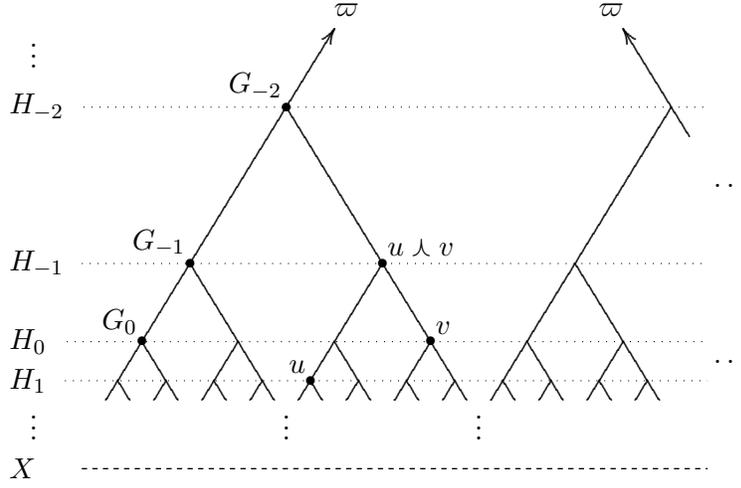
FIGURE 1. Tree of balls $\mathcal{T}(X)$ with forward degree $n_l = 2$.

$\mathcal{H}(T)$ are orthogonal. It follows that the eigenspace $\mathcal{H}_k$ corresponding to the horocycle H_k (equivalently, to the eigenvalue λ_k) is of the form

$$\mathcal{H}_k = \bigoplus_{B \in H_k} \mathcal{H}(B).$$

The system of eigenfunctions $\{f_{B_i,B}\}_{B \in \mathcal{B}}$ is complete, whence

$$\bigoplus_{k \in \mathbb{Z}} \mathcal{H}_k = L^2(G, m).$$

Among the variety of homogeneous hierarchical Laplacians L_C on (G, d, m), we would like to mention a one-parameter family $\{\mathfrak{B}^\alpha\}_{\alpha > 0}$. The hierarchical Laplacian $\mathfrak{B}^\alpha$ is defined by the function

$$C^\alpha(B) = (m(B))^{-\alpha} - \left(m(B')\right)^{-\alpha}, \tag{2.2}$$

where $B \subset B'$ are nearest neighbor balls. Thus for any $B \in \mathcal{B}$, the eigenvalue $\lambda^\alpha(B)$ of $\mathfrak{B}^\alpha$ corresponding to B is

$$\lambda^\alpha(B) = (m(B))^{-\alpha}.$$

We recall from [9] that the set $\mathcal{D}$ of compactly supported locally constant functions is in the domain of the operator $\mathfrak{B}^\alpha$. It is remarkable, although not difficult to prove, that the following properties hold:

$$\mathfrak{B}^\beta : \mathcal{D} \to \mathsf{Dom}(\mathfrak{B}^\alpha),$$

and on $\mathcal{D}$,

$$\mathfrak{B}^\alpha \circ \mathfrak{B}^\beta = \mathfrak{B}^{\alpha+\beta} \text{ and } (\mathfrak{B}^\alpha)^\beta = \mathfrak{B}^{\alpha\beta}.$$

Moreover, when $G = \mathbb{Q}_p$, the operator $\mathfrak{D}^\alpha = p^\alpha \mathfrak{B}^\alpha$ is the fractional derivative operator of order α as defined and studied via Fourier transform in [38], [39], [41] and [21]:

$$\widehat{\mathfrak{D}^\alpha f}(\xi) = |\xi|^\alpha \, \widehat{f}(\xi), \quad \xi \in \mathbb{Q}_p,$$

and

$$\mathfrak{D}^\alpha f(x) = \frac{p^\alpha - 1}{1 - p^{-\alpha-1}} \int_G \frac{f(x) - f(y)}{|x - y|^{1+\alpha}} dm(y). \tag{2.3}$$

Note that similar identifications (based on cyclic groups $\mathbb{Z}(n)$ as building blocks) can be carried over when (X, d) is infinite and discrete. For instance, the infinite (non-Abelian) symmetric group S_∞ equipped with its canonical ultrametric structure defined by the family $\{S_n\}$ of its finite symmetric subgroups can be identified (as an ultrametric measure space) with the discrete Abelian group $G = \bigoplus_{\ell > 1} \mathbb{Z}(\ell)$. The group G is equipped with its canonical ultrametric structure defined by the family $\{G_n\}$ of its finite subgroups $G_n = \prod_{\ell=2}^n \mathbb{Z}(\ell)$.

3. Random perturbations

Let (X, d, m) be a non-compact homogeneous ultrametric space. Let L_C be the homogeneous hierarchical Laplacian acting on X and defined by the function C. Let $\{\varepsilon(B)\}_{B \in \mathcal{B}}$ be a family of symmetric i.i.d. random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and taking values in some small interval $[-\epsilon, \epsilon] \subset (-1, 1)$. We define the perturbed function $\mathcal{C}$ and the perturbed hierarchical Laplacian $\mathcal{L}$ as

$$\mathcal{C}(B) = C(B)(1 + \varepsilon(B)),$$

and

$$\mathcal{L}f(x) = \sum_{B \in \mathcal{B}: \, x \in B} \mathcal{C}(B) \left(f(x) - \frac{1}{m(B)} \int_B f \, dm \right).$$

Evidently, $\mathcal{L}(\omega)$ may well be non-homogeneous for some $\omega \in \Omega$. Still it has a pure point spectrum for all ω, but the structure of the closed set $\mathsf{Spec}(\mathcal{L}(\omega))$ can be quite complicated. See [10] for various examples.

Two stationary families.

We study the eigenvalues of the perturbed operator. Because of the homogeneity assumption we can identify its eignevalues with a certain stationary family of random variables. This observation is crucial in our analysis.

Let us fix a horocycle $H = H_\ell$ for some $\ell \in \mathbb{Z}$ and let $\lambda_H = \lambda_\ell$ be the eigenvalue of the homogeneous Laplacian L_C corresponding H. For a given $B \in H$, let $\{B_k\}_{k \leq \ell}$ be the unique geodesic path in $\mathcal{T}(X)$ from ϖ to B. The eigenvalue $\lambda(B)$ of the perturbed operator $\mathcal{L}$ is

$$
\begin{aligned}
\lambda(B) &= \sum_{k \leq \ell} \mathcal{C}(B_k) = \sum_{k \leq \ell} c_k \left(1 + \varepsilon(B_k) \right) \\
&= \lambda_\ell \left(1 + \sum_{k \leq \ell} \alpha_k \varepsilon(B_k) \right) := \lambda_\ell \left(1 + U(B) \right),
\end{aligned}
$$

where $\alpha_k = c_k/\lambda_\ell$, and

$$U(B) = \sum_{k \leq \ell} \alpha_k \varepsilon(B_k). \tag{3.1}$$

Notice that $\sum_{k \leq \ell} \alpha_k = 1$ and that $\{U(B)\}_{B \in H}$ are (dependent) identically distributed symmetric random variables taking values in some symmetric interval $I \subset (-1, 1)$.

As the horocycle $H = H_\ell$ is fixed, it is useful to identify the balls $B \in H$ with elements $g \in G$ of the (discrete!) Abelian group $G = \bigoplus_{k<\ell} \mathbb{Z}(n_k)$. With such an identification in mind it is now straightforward to show that the family of random variables $\{U(g)\}_{g \in G} = \{U(B)\}_{B \in H}$ (respectively, $\{\lambda(g)\}_{g \in G} = \{\lambda(B)\}_{B \in H}$) is stationary. That is, for any $g, g_1, ..., g_s$ in G,

$$\{U(g + g_1), \ldots, U(g + g_s)\} \overset{d}{=} \{U(g_1), \ldots, U(g_s)\},$$

respectively,

$$\{\lambda(g + g_1), \ldots, \lambda(g + g_s)\} \overset{d}{=} \{\lambda(g_1), \ldots, \lambda(g_s)\}.$$

For the general theory of stationary processes indexed by $\mathbb{Z}$ or $\mathbb{R}$ we refer to [20] and [28].

The integrated density of states.

Let $O \in \mathcal{B}$ be an ultrametric ball. We fix a horocycle H and define the normalized empirical process

$$\mathcal{M}_{HO} = \frac{1}{|\mathcal{B}_{HO}|} \sum_{B \in \mathcal{B}_{HO}} \delta_{\lambda(B)}, \tag{3.2}$$

where $\mathcal{B}_{HO} = \{B \in \mathcal{B} : B \in H \text{ and } B \subset O\}$. As was shown in [7], there exists a probability measure $\mathcal{M}_H$ such that $\mathbb{P}$-a.s.

$$\mathcal{M}_{HO} \to \mathcal{M}_H \text{ weakly as } O \to \varpi. \tag{3.3}$$

The measure $\mathcal{M}_H$ is called the *integrated density of states* (corresponding to the horocycle H). If there exists a measurable function $\mathfrak{m}_H : \mathbb{R} \to \mathbb{R}$ such that for any interval I

$$\mathcal{M}_H(I) = \int_I \mathfrak{m}_H(\tau) d\tau, \tag{3.4}$$

the function $\mathfrak{m}_H(\tau)$ is called the density of states (corresponding to the horocycle H). The question of whether $\mathfrak{m}_H$ exists, is continuous, belongs to the class C^∞, etc... is basic in various applications; see for instance (4.5) in Section 4.

Recall that for any $B \in H$,

$$\lambda(B) = \lambda_H \left(1 + U(B)\right),$$

where the $U(B)$ are identically distributed (dependent) symmetric random variables. Let $\mathcal{N}_H$ denote their common distribution. It turns out that the normalized empirical process

$$\mathcal{N}_{HO} = \frac{1}{|\mathcal{B}_{HO}|} \sum_{B \in \mathcal{B}_{HO}} \delta_{U(B)}$$

converges to $\mathcal{N}_H$ weakly $\mathbb{P}$-a.s. Evidently, the probability measures $\mathcal{M}_H$ and $\mathcal{N}_H$ are related by the affine transformation

$$\mathcal{N}_H = \mathcal{M}_H \circ \vartheta, \quad \vartheta : \tau \to \lambda_H \tau + \lambda_H.$$

In particular, $\mathcal{M}_H$ is absolutely continuous w.r.t. Lebesgue measure if and only if $\mathcal{N}_H$ is. The measure $\mathcal{N}_H$ has a remarkable feature – it belongs to the class $\mathfrak{J}$ of probability measures which can be represented as the distribution of some random variable U of the form

$$U = \sum_{k \geq 0} b_k \varepsilon_k, \tag{3.5}$$

where $\{\varepsilon_k\}_{k \geq 0}$ are symmetric i.i.d. random variables and $b_k > 0$ satisfies $\sum b_k = 1$.

Various properties of $\mathfrak{J}$-distributions (infinite convolutions) have been studied by many authors since the 1930's. See e.g. [29], [19], [37], [34], [33], [36] and references therein. We would like to mention three remarkable properties of $\mathfrak{J}$-distributions. The first one is due to Lévy (1937) and the second due to Jessen and Wintner (1935), see e.g. [29], Thm. 3.7.6 and 3.7.7 respectively. For the third property, we refer to [7].

- Each $\mathfrak{J}$-distribution $\mathcal{N}$ in its Lebesgue decomposition contains no discrete component.
- Assume that the $\{\varepsilon_k\}$ from (3.5) are discrete. Then the measure $\mathcal{N}$ is either singular or it is absolutely continuous w.r.t. Lebesgue measure.
- Assume that the common characteristic function ϕ of the i.i.d. $\{\varepsilon_k\}$ tends to zero at infinity and that the coefficients satisfy $b_k \geq C \exp(-Dk)$, for some $C, D > 0$ and all k. Then $\mathcal{N}$ admits a C^∞-density w.r.t. Lebesgue measure.

Various examples of singular random variables ε having characteristic functions ϕ which tend to zero at infinity are given in Section 3 of [29] and also in Sections 6 and 7 of [19]. Here is an example due to Kerschner (1936): Suppose a is a rational number such that $0 < a < 1/2$, and $a \neq 1/n$ for any integer $n \geq 3$. Then

$$\phi(x) = \prod_{k=1}^{\infty} \cos\left(a^k x\right)$$

is the characteristic function of a singular symmetric $\mathfrak{J}$-distribution satisfying

$$|\phi(x)| \leq \frac{c}{(\log |x|)^\gamma} \text{ at } \infty \tag{3.6}$$

for some $\gamma, c > 0$.

4. The Poisson Convergence

Fix a horocycle H. The eigenvalues $\lambda(B)$ can be represented by the empirical process

$$M_O = \sum_{B \in \mathcal{B}_{HO}} \delta_{\lambda(B)}.$$

The *intensity measure* $\mu_O(I)$ gives the expected number of $\lambda(B)$, $B \in \mathcal{B}_{HO}$, which fall in the interval I. It is computed as

$$\mu_O(I) = \mathbb{E}[M_O(I)] = |\mathcal{B}_{HO}| \, \mathbb{P}\left\{\lambda(B) \in I\right\}.$$

Recall that the right-hand side of the above equality does not depend on $B \in \mathcal{B}_{HO}$. We fix numbers $c, \tau_0 > 0$ and consider a small interval

$$I_O = \left\{\tau : |\tau - \tau_0| \leq \frac{c}{2\,|\mathcal{B}_{HO}|}\right\}. \tag{4.1}$$

Assuming that the density of states $\mathfrak{m}(\tau) := \mathfrak{m}_H(\tau)$ (as defined in (3.4)) exists and is continuous at τ_0, and that $\mathfrak{m}(\tau_0) > 0$, we obtain

$$\lim_{O \to \varpi} \mu_O(I_O) = c\mathfrak{m}(\tau_0) =: \lambda_c > 0. \tag{4.2}$$

If the $\lambda(B)$, $B \in \mathcal{B}_{HO}$, were i.i.d., then (4.2) would yield the classical convergence of the random variables $M_O(I_O)$ to the Poisson distribution $\mathrm{Poi}(\lambda)$ with intensity $\lambda := \lambda_c$. More precisely, in the i.i.d. case we would then have (see [28], [13], [4])

$$\|\mathscr{L}(M_O(I_O)) - \mathrm{Poi}(\lambda_O)\|_{TV} \leq \frac{\min(\lambda_O, \lambda_O^2)}{2\,|\mathcal{B}_{HO}|}$$

where $\lambda_O = \mu_O(I_O) \to \lambda$ as $O \to \varpi$. However, in our case the $\lambda(B)$, $B \in \mathcal{B}_{HO}$, are *dependent* random variables, whence the classical theory does not apply directly and needs some justifications and complements.

In the course of our study we assume that

$$1/\kappa \leq C(B)m(B)^\delta \leq \kappa, \tag{4.3}$$

equivalently,

$$1/2\kappa \leq \lambda(B)m(B)^\delta \leq 2\kappa, \tag{4.4}$$

holds for all $B \in \mathcal{B}$ and some $\delta > 1, \kappa > 0$. Assume also that the common law $\mathscr{L}(\varepsilon)$ of i.i.d. $\varepsilon(B)$ admits a bounded density. Then it was shown in [7] that

$$\mathscr{L}(M_O(I_O)) \to \mathrm{Poi}(\lambda) \text{ weakly as } O \to \varpi. \tag{4.5}$$

Examples provided in [7] show that the assumption that $\mathscr{L}(\varepsilon)$ admits a bounded density can not be dropped entirely. Whether the assumption that (4.4) holds with $\delta > 1$ can be relaxed is very much open at present writing.

We aim to estimate the rate of convergence of $\mathscr{L}(M_O(I_O)) \to \mathrm{Poi}(\lambda)$ in total variation. To this end, we apply an approximation result from [4]. Recall that for any $B \in H$, the eigenvalue $\lambda(B)$ can be written in the form

$$\lambda(B) = \lambda_H(1 + U(B)) \tag{4.6}$$

where

$$U(B) = \sum_{B \subseteq B_k} \alpha_k \varepsilon(B_k) \tag{4.7}$$

and $\alpha_k = C(B_k)/\lambda_H$. The common distribution function $\mathcal{N}_H$ of the family $\{U(B)\}_{B \in H}$ is absolutely continuous and its density $\mathfrak{n}_H$ relates to the integrated density of states $\mathfrak{m}_H$ by the equation

$$\mathfrak{n}_H(t) = \lambda_H \mathfrak{m}_H(\lambda_H t + \lambda_H).$$

In particular, $\mathfrak{n}_H$ is supported by some interval $[-\epsilon, \epsilon] \subset (-1, 1)$, is continuous, and is strictly positive at $t_0 = \tau_0/\lambda_H - 1$. Let N_O be the empirical process defined by the family $\{U(B)\}$, $B \in \mathcal{B}_{HO}$, and choose the interval J_O as

$$J_O = \left\{ t : |t - t_0| \leq \frac{d}{2 |\mathcal{B}_{HO}|} \right\}, \ d = c/\lambda_H.$$

Then equation (4.6) yields

$$\mathbb{P}\{M_O(I_O) = k\} = \mathbb{P}\{N_O(J_O) = k\}.$$

Thus in what follows, we can restrict our attention to the family $\{U(B)\}_{B \in H}$. Since we fix a horocycle H we can further simplify our task by replacing X with the discrete Abelian group

$$G = \bigoplus_{k \geq 1} \mathbb{Z}(n_k), \ n_k \in \mathbb{N}_{\geq 2},$$

equipped with its canonical ultrametric structure defined by the family $\{G_\ell\}_{\ell \geq 0}$ of its finite subgroups

$$G_0 = \{e\}, \ G_\ell = \prod_{1 \leq k \leq \ell} \mathbb{Z}(n_k).$$

Denote the order of G_ℓ by π_ℓ so that $\pi_\ell = n_0 n_1 \cdots n_\ell$ with the convention that $n_0 = 1$. We have that $H = H_0$ is the set of all singletons, H_1 is the set of all ultrametric balls of the form $g + G_1$, and so forth.

When $B = \{g\}$ we write $U(B) = U_g(\omega)$. For the increasing sequence of balls $B_k \supseteq B$, we set $\varepsilon(B_k) = \varepsilon(g_k)$. We assume henceforth that the random variables $\{\varepsilon(g_k)\}_{g \in G, k \in \mathbb{N}_0}$ are independent and satisfy

- $|\varepsilon(g_k)| < 1$ for all $g \in G$, $k \geq 0$.
- $\{\varepsilon(g_k)\}_{g \in G}$ are identically distributed for each $k \geq 0$.
- $\varepsilon(e_0)$ is absolutely continuous with bounded density η_ε.

In particular, there is no symmetry assumption, $\varepsilon(g_k)$ and $\varepsilon(g_\ell)$ do not have to have the same distribution for $k \neq \ell$, and $\varepsilon(g_k)$ does not have to be absolutely continuous (or have bounded density) for $k > 0$. Though these relaxations are not all applicable to the setting of perturbed hierarchical Laplacians as previously defined, it is of interest to prove a quantitative law of rare events with the same basic dependence structure in as much generality as possible.

Define the stationary process $\{U_g\}_{g \in G}$ by

$$U_g = \sum_{k=0}^{\infty} \alpha_k \varepsilon_{g_k}.$$

In what follows, we need only assume that the sequence $\{\alpha_k\}_{k=0}^{\infty}$ satisfies

$$\sum_{k=\ell}^{\infty} \alpha_k \leq K \pi_\ell^{-(1+\gamma)}$$

for some constants $K, \gamma > 0$. Here we are thinking of $\gamma = \delta - 1$, a more convenient notation for subsequent computations. Note that absolute continuity of $\varepsilon(g_0)$ ensures that U_g is absolutely continuous. Moreover, writing $U_g = Y_g^0 + T_g^0$ where $Y_g^0 = \alpha_0 \varepsilon(g_0)$

has density $f_0(x) = \frac{1}{\alpha_0}\eta_\varepsilon\left(\frac{x}{\alpha_0}\right)$ and $T_g^0 = \sum_{k=1}^{\infty}\alpha_k\varepsilon(g_k)$ has distribution ν_0, we see that the density of U_g satisfies

$$\eta(x) = \int f_0(x-y)d\nu_0(y) \le \|f_0\|_\infty \int d\nu_0(y) = \frac{1}{\alpha_0}\|\eta_\varepsilon\|_\infty \qquad (4.8)$$

for all $x \in \mathbb{R}$. (We write η rather than $\mathfrak{n}$ to emphasize that we are working with slightly weaker assumptions than are used in the proof of (4.5) from [7].)

Let t_0 be a Lebesgue point of η such that $\eta(t_0) > 0$ and let c be some fixed positive number. Set $I_\ell = \left[t_0 - \frac{c}{2\pi_\ell}, t_0 + \frac{c}{2\pi_\ell}\right]$, and define

$$X_g^\ell = 1\{U_g \in I_\ell\}, \ W_\ell = \sum_{g \in G_\ell} X_g^\ell.$$

We are interested in providing error terms for the Poisson approximation of W_ℓ, the analogue of $N_O(J_O)$.

For each $\ell \ge 0$, the X_g^ℓ's are identically distributed, so if $p_\ell = \mathbb{P}\{U_e \in I_\ell\}$, then $\mathbb{E}[W_\ell] = \pi_\ell p_\ell$. Writing $\lambda(\ell)$ for this expectation, it follows from the Lebesgue differentiation theorem that

$$\lim_{\ell\to\infty}\lambda(\ell) = c\lim_{\ell\to\infty}\frac{\pi_\ell}{c}\int_{t_0-\frac{c}{2\pi_\ell}}^{t_0+\frac{c}{2\pi_\ell}}\eta(x)dx = c\eta(t_0) =: \lambda.$$

We now state our main result, which proves a quantitative version of the convergence in (4.5).

Theorem 4.1. *Keeping the above assumptions, set* $m = \inf_{j\ge 1} n_j$, $M = \sup_{j\ge 1} n_j$,

$$C = \frac{1-e^{-\lambda(\ell)}}{\lambda(\ell)}\left(\lambda(\ell)^2 + \frac{c^2}{\alpha_0^2}\|\eta_\varepsilon\|_\infty^2\right) + \frac{16K}{\alpha_0}\left(1 \wedge \lambda(\ell)^{-\frac{1}{2}}\right)\|\eta_\varepsilon\|_\infty,$$

and let M_ℓ be the running maximum of the subsequence of $\{n_j\}_{j=1}^{\infty}$ defined in (5.3).
(a) If $M < \infty$, then

$$\|\mathscr{L}(W_\ell) - \mathrm{Poi}(\lambda(\ell))\|_{TV} \le Cm^{-\ell\frac{\gamma}{\log_m(M)+1+\gamma}}.$$

(b) If $M = \infty$, then

$$\|\mathscr{L}(W_\ell) - \mathrm{Poi}(\lambda(\ell))\|_{TV} \le CM_{\ell-1}^{-\gamma/3}.$$

Note that $\dfrac{1-e^{-x}}{x} \le 1 \wedge x^{-1}$ and

$$\lambda(\ell) = \pi_\ell\int_{I_\ell}\eta(x)dx \le \pi_\ell|I_\ell|\|\eta\|_\infty \le \frac{c\|\eta_\varepsilon\|_\infty}{\alpha_0},$$

so we have

$$C \le \lambda(\ell) + \frac{c^2\|\eta_\varepsilon\|_\infty^2}{\alpha_0^2} + \frac{16K\|\eta_\varepsilon\|_\infty}{\alpha_0} \le \frac{\|\eta_\varepsilon\|_\infty}{\alpha_0}\left(c + \frac{c^2\|\eta_\varepsilon\|_\infty}{\alpha_0} + 16K\right), \qquad (4.9)$$

which is independent of both t_0 and ℓ.

With Theorem 4.1 in hand, the rate of convergence to the Poi(λ) distribution can be estimated as

$$\|\mathscr{L}\left(W_\ell\right) - \text{Poi}(\lambda)\|_{TV} \leq \|\mathscr{L}\left(W_\ell\right) - \text{Poi}\left(\lambda(\ell)\right)\|_{TV} + \|\text{Poi}\left(\lambda(\ell)\right) - \text{Poi}(\lambda)\|_{TV}$$

$$= \|\mathscr{L}\left(W_\ell\right) - \text{Poi}\left(\lambda(\ell)\right)\|_{TV} + \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{k!}\left|\lambda(\ell)^k e^{-\lambda(\ell)} - \lambda^k e^{-\lambda}\right|,$$

which goes to zero since $\lambda(\ell) \to \lambda$. See [1] for more concise bounds on the second term in the final expression.

The next section is devoted to the proof of Theorem 4.1, but first we consider a concrete application of this result. Let $\mathbb{Q}_p$ be the field of p-adic numbers and let $\mathfrak{D}^\alpha$ be the fractional derivative operator of order α defined in (2.3); see also (2.2). Then $\mathfrak{D}^\alpha$ is a hierarchical Laplacian corresponding to the function $C(B) = (p^\alpha - 1)m(B)^{-\alpha}$, so its eigenvalues are $\lambda(B) = p^\alpha m(B)^{-\alpha}$. Every p-adic ball in the horocycle H_k is of the form $B = a + p^k\mathbb{Z}_p$ (where $\mathbb{Z}_p$ is the compact group of p-adic integers) and thus has Haar measure $m(B) = p^{-k}$. Therefore every $B \in H_k$ gives rise to the eigenvalue $\lambda_k = p^{\alpha(k+1)}$.

Without loss of generality, we fix the horocycle $H = H_{-1}$ corresponding to the eigenvalue $\lambda_{-1} = 1$. We consider a perturbation of $\mathfrak{D}^\alpha$ by i.i.d. $\varepsilon(B) \sim \text{Unif}(-1, 1)$. Since H is fixed, we can replace $\mathbb{Q}_p$ with $G = \bigoplus_{k \geq 1}\mathbb{Z}(n_k)$ where $n_k = p$ for all k. One readily checks that (4.4) is satisfied with $\delta = \alpha$ and $\alpha > 1$. Setting $\alpha_k = C(B_{-1-k})/\lambda_{-1} = (1 - p^{-\alpha})p^{-\alpha k}$, we have

$$\sum_{k=\ell}^{\infty}\alpha_k = (1 - p^{-\alpha})\sum_{k=\ell}^{\infty}p^{-\alpha k} = p^{-\alpha\ell} = K\pi_\ell^{-(1+\gamma)}$$

with $\gamma = \alpha - 1$, $K = 1$.

For the sake of notational simplicity, we fix $t_0 = 0$ and $c = \pi$. We have that $U = U_e = \sum_{k=0}^{\infty}\alpha_k\varepsilon_k$ with $\{\varepsilon_k\}_{k=0}^{\infty}$ i.i.d. $\text{Unif}(-1, 1)$, so U has characteristic function $\varphi(t) = \prod_{k=0}^{\infty}\frac{\sin(\alpha_k t)}{\alpha_k t}$ (where we take $\frac{\sin(0)}{0} = 1$). Fourier inversion [16] gives

$$\lambda = \pi\eta(0) = \frac{1}{2}\int_{-\infty}^{\infty}\prod_{k=0}^{\infty}\frac{\sin(\alpha_k s)}{\alpha_k s}ds$$

and

$$\lambda(\ell) = p^\ell\mathbb{P}\left\{-\frac{\pi}{2p^\ell} \leq U \leq \frac{\pi}{2p^\ell}\right\}$$

$$= \frac{p^\ell}{2\pi}\int_{-\infty}^{\infty}\frac{\exp\left(\frac{\pi is}{2p^\ell}\right) - \exp\left(-\frac{\pi is}{2p^\ell}\right)}{is}\varphi(s)ds$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\frac{2p^\ell}{\pi s}\sin\left(\frac{\pi s}{2p^\ell}\right)\prod_{k=0}^{\infty}\frac{\sin(\alpha_k s)}{\alpha_k s}ds.$$

Since $\|\eta_\varepsilon\|_\infty = \frac{1}{2}$, $m = M = p$, and $\alpha_0 = 1 - p^{-\alpha}$, Theorem 4.1 shows that

$$\|\mathscr{L}(W_\ell) - \text{Poi}(\lambda(\ell))\|_{TV} \leq Cp^{-\frac{\alpha-1}{\alpha+1}\ell}$$

where, by Equation (4.9),

$$C \leq \frac{1}{2(1-p^{-\alpha})}\left(\pi + \frac{\pi^2}{2(1-p^{-\alpha})} + 16\right) < 30.$$

In particular, the bound on the error is of order

$$p^{-\frac{\alpha-1}{\alpha+1}\ell} = \left(\frac{1}{|B_{-\ell}|}\right)^{\frac{\alpha-1}{\alpha+1}},$$

whereas one would expect a $O\left(\frac{1}{|B_{-\ell}|}\right)$ bound if the $\lambda(B)$ terms were independent.

5. Convergence Rates

To prove Theorem 4.1, we appeal to the following result from [4].

Theorem 5.1. *Suppose that $\{X_\alpha\}_{\alpha \in J}$ are Bernoulli random variables with success probabilities $p_\alpha = \mathbb{P}\{X_\alpha = 1\} \in (0,1)$. For $\alpha \neq \beta$, write $p_{\alpha\beta} = \mathbb{E}[X_\alpha X_\beta]$. Set $W = \sum_{\alpha \in J} X_\alpha$ and assume that $\mu = \mathbb{E}[W] < \infty$. For each $\alpha \in J$, let $B_\alpha \subseteq J$ be a subset containing α. Define $V_\alpha = \sum_{\beta \in J \setminus B_\alpha} X_\beta$,*

$$b_1 = \sum_{\alpha \in J}\sum_{\beta \in B_\alpha} p_\alpha p_\beta,$$

$$b_2 = \sum_{\alpha \in J}\sum_{\beta \in B_\alpha \setminus \{\alpha\}} p_{\alpha\beta},$$

$$b_3 = \sup_{f \in \mathcal{F}}\sum_{\alpha \in J} \mathbb{E}[(X_\alpha - p_\alpha)f(V_\alpha + 1)],$$

where $\mathcal{F} = \left\{f : \mathbb{N}_0 \to \mathbb{R} \text{ such that } \|f\|_\infty \leq 1 \wedge \mu^{-\frac{1}{2}}\right\}$. Then

$$\|\mathscr{L}(W) - \text{Poi}(\mu)\|_{TV} \leq \frac{1-e^{-\mu}}{\mu}(b_1 + b_2) + b_3.$$

For the problem considered here, we have $J = G_\ell$, $X_\alpha = X_g^\ell$, $\mu = \lambda(\ell)$. We take our *dependency neighborhoods* to be the closed k-balls

$$B_g = g + G_k \tag{5.1}$$

for some $k = k(\ell) \in [0,\ell)$ to be determined. The following lemma provides bounds on b_1, b_2 and b_3 in terms of the size of the dependency neighborhoods B_g.

Lemma 5.2. *Let b_1, b_2 and b_3 be the constants introduced in Theorem 5.1 where the other terms are interpreted as in the preceding paragraph. Then*

$$b_1 = \lambda(\ell)^2 \pi_k \pi_\ell^{-1},$$

$$b_2 \leq \frac{c^2}{\alpha_0^2}\|\eta_\varepsilon\|_\infty^2 \pi_k \pi_\ell^{-1},$$

$$b_3 \leq \frac{16K}{\alpha_0}\left(1 \wedge \lambda(\ell)^{-\frac{1}{2}}\right)\|\eta_\varepsilon\|_\infty \pi_\ell \pi_{k+1}^{-(1+\gamma)}.$$

The proof of Lemma 5.2 is postponed until Subsection 5. For the moment, note that the dependency neighborhoods (5.1) must be chosen with care. If k is too large, then the b_1 and b_2 estimates do not go to zero as $\ell \to \infty$. If it is too small, then the bound on b_3 does not go to zero. Also, it is worth observing that in most known applications of Theorem 5.1, the dependency neighborhoods can be chosen so that $b_3 = 0$. When there is long-range dependence, it is typically more efficient to get error bounds using size-bias couplings [5, 6, 35]. However, in the case at hand, it seems that Theorem 5.1 is the most straightforward approach, the difficulty of bounding b_3 notwithstanding.

The next lemma guarantees that the dependency neighborhoods can always be chosen to make the bounds in Lemma 5.2 vanish as $\ell \to \infty$. To state it, we introduce some notation. Recall the sequence $\{n_j\}_{j=1}^{\infty} \subseteq \mathbb{N}_{\geq 2}$ defining G and the choice of $n_0 = 1$. Set

$$m = \inf_{j \geq 1} n_j, \quad M = \sup_{j \geq 1} n_j. \tag{5.2}$$

When $M = \infty$, define $\{n_{j_i}\}_{i=0}^{\infty} \subseteq \{n_j\}_{j=0}^{\infty}$ to be the subsequence where

$$j_0 = 0 \quad \text{and} \quad j_{i+1} = \min\{j > j_i : n_j > n_{j_i} \text{ and } n_j > n_{j_i}^{\gamma/3}\}, \tag{5.3}$$

and observe that since $n_{j_0} = n_0 = 1$, we have $j_1 = 1$. For $\ell \geq 0$ we define

$$M_\ell = \sup\{n_{j_i} : j_i \leq \ell\}. \tag{5.4}$$

We now have the following result.

Lemma 5.3. *Let m, M, and M_ℓ be defined as in (5.2) and (5.4), respectively. Then for every $\ell \geq 1$, we can always choose $k = k(\ell) < \ell$ such that*
(a) If $M < \infty$, then

$$\pi_k \pi_\ell^{-1}, \pi_\ell \pi_{k+1}^{-(1+\gamma)} \leq m^{-\ell \frac{\gamma}{\log_m(M)+1+\gamma}}.$$

(b) If $M = \infty$, then

$$\pi_k \pi_\ell^{-1}, \pi_\ell \pi_{k+1}^{-(1+\gamma)} \leq M_{\ell-1}^{-\gamma/3}.$$

Observe that Theorem 4.1 is now a direct consequence of Theorem 5.1, together with Lemmas 5.2 and 5.3. The rest of the paper is devoted to proving the two lemmas.

Proof of Lemma 5.3. Suppose that $M < \infty$. Then

$$\pi_\ell \pi_{k+1}^{-(1+\gamma)} = \frac{n_{k+2} \times \ldots \times n_\ell}{(n_1 \times \ldots \times n_{k+1})^\gamma} \leq M^{\ell-(k+1)} m^{-\gamma(k+1)} = m^{\ell \log_m(M) - (k+1)(\log_m(M)+\gamma)}$$

and

$$\pi_k \pi_\ell^{-1} = \frac{1}{n_{k+1} \times \ldots \times n_\ell} \leq m^{k-\ell}.$$

Choosing

$$k = \left\lfloor \frac{\ell \left(\log_m(M) + 1\right)}{\log_m(M) + 1 + \gamma} \right\rfloor$$

and using $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$ completes the proof of part (a).

Part (b) is much more involved. To prove it, we specify a rule for picking $k = k(\ell)$ for every $\ell \geq 1$. Suppose that $M = \infty$ and recall the subsequence $\{n_{j_i}\}_{i=0}^{\infty}$ introduced in (5.3). When $\ell = 1$, we choose $k(1) = 0$, and since $M_0 = 1$, we see that

$$\pi_k \pi_\ell^{-1} = \pi_0 \pi_1^{-1} = \frac{1}{n_1} \leq M_0^{-\gamma/3} \quad \text{and} \quad \pi_\ell \pi_{k+1}^{-(1+\gamma)} = \pi_1 \pi_1^{-(1+\gamma)} = \frac{1}{n_1^\gamma} \leq M_0^{-\gamma/3}.$$

Fix $i \geq 1$. We now outline a procedure to choose k when $\ell \in (j_i, j_{i+1}]$ such that

$$\pi_k \pi_\ell^{-1}, \pi_\ell \pi_{k+1}^{-(1+\gamma)} \leq n_{j_i}^{-\gamma/3}. \tag{5.5}$$

Since $j_1 = 1$, this procedure will specify $k(\ell)$ for all $\ell \geq 2$, and the result will follow since $\ell \in (j_i, j_{i+1}]$ if and only if $M_{\ell-1} = n_{j_i}$.

We first define

$$B = \{b \in (j_i, j_{i+1}] : n_b \geq n_{j_i}^{\gamma/3}\} \quad \text{and} \quad S = (j_i, j_{i+1}] \setminus B.$$

The sets B and S should be interpreted as sets of "big" and "small" indices, respectively. For any integer $r \in [j_i - 1, j_{i+1} - 2]$, define the integers $u(r)$ and $v(r)$ by

$$u(r) = \min \left\{ u \in [r+2, j_{i+1}] : u \in B \text{ or } \prod_{j=r+2}^{u} n_j \geq n_{j_i}^{2\gamma/3} \right\}$$

and

$$v(r) + 1 = \begin{cases} \min \left\{ v \in [r+2, j_{i+1}] : \prod_{j=r+2}^{v} n_j \geq n_{j_i}^{\gamma/3} \right\}, & u(r) \in S, \\ u(r), & u(r) \in B. \end{cases}$$

We see that these quantities are always well defined, because $j_{i+1} \in B$ by (5.3).

One may interpret $u(r)$ and $v(r)$ as follows: Suppose $\ell_0 \in (j_i, j_{i+1})$ and $k_0 = k(\ell_0) < \ell_0$ satisfies (5.5). Then $u(k_0)$ tells us how far we can increase ℓ past ℓ_0 while keeping $k(\ell)$ fixed at k_0 and not violating (5.5). For example, suppose we keep $k(u(k_0)) = k_0$ once ℓ reaches $u(k_0)$, then we are no longer guaranteed that $\pi_{u(k_0)} \pi_{k_0+1}^{-(1+\gamma)} \leq n_{j_i}^{-\gamma/3}$. We see that $k(u(k_0))$ must be increased past k_0, but it cannot be increased too much, because we must maintain $\pi_{k(u(k_0))} \pi_{u(k_0)}^{-1} \leq n_{j_i}^{-\gamma/3}$. Hence, we use $v(u(k_0))$ to tell us exactly how far to increase $k(u(k_0))$. We then repeat this process inductively.

We now formalize the discussion above. For any $\ell \in [j_i, j_{i+1}]$, we set

$$k(\ell) = \begin{cases} k(w_{m-1}), & w_{m-1} \leq \ell < w_m, \\ v(k(w_{m-1})), & \ell = w_m, \end{cases}$$

where

$$w_0 = j_i, \quad k(w_0) = j_i - 1 \quad \text{and} \quad w_m = u(k(w_{m-1}))$$

for all $m \geq 1$ such that $w_m \leq j_{i+1}$.

We now verify that choosing $k(\ell)$ according to this procedure satisfies (5.5) for all $\ell \in (j_i, j_{i+1}]$. Observe that when $u(r) \in S$, then

$$\prod_{j=r+2}^{u(r)-1} n_j < n_{j_i}^{2\gamma/3}, \quad \prod_{j=r+2}^{u(r)} n_j < n_{j_i}^{\gamma}, \tag{5.6}$$

and

$$\prod_{j=r+2}^{v(r)} n_j < n_{j_i}^{\gamma/3}, \tag{5.7}$$

with the convention that the empty product equals 1. For $m \geq 1$, if $\ell = w_m \in B$, then $k(\ell) = \ell - 1$ and

$$\pi_\ell \pi_{k+1}^{-(1+\gamma)} = \pi_\ell^{-\gamma} \leq n_{j_i}^{-\gamma/3} \quad \text{and} \quad \pi_k \pi_\ell^{-1} = n_{w_m}^{-1} \leq n_{j_i}^{-\gamma/3}.$$

Next, suppose that $\ell = w_m \in S$ for some $m \geq 1$ and $w_m < j_{i+1}$. Then using (5.6) together with the definition $v(r)$ and the fact that $j_i \leq k(w_{m-1}) + 1$, we see that

$$\pi_\ell \pi_{k+1}^{-(1+\gamma)} = \pi_{w_m} \pi_{k(w_m)+1}^{-(1+\gamma)}$$
$$= \left(\prod_{j=1}^{k(w_{m-1})+1} n_j^{-\gamma} \right) \left(\prod_{j=k(w_{m-1})+2}^{u(k(w_{m-1}))} n_j \right) \left(\prod_{j=k(w_{m-1})+2}^{v(k(w_{m-1}))+1} n_j^{-(1+\gamma)} \right)$$
$$< n_{j_i}^{-\gamma} n_{j_i}^{\gamma} n_{j_i}^{-\gamma/3} = n_{j_i}^{-\gamma/3}.$$

Furthermore, (5.7) implies

$$\pi_k \pi_\ell^{-1} = \pi_{k(w_m)} \pi_{w_m}^{-1}$$
$$= \left(\prod_{j=k(w_{m-1})+2}^{v(k(w_{m-1}))} n_j \right) \left(\prod_{j=k(w_{m-1})+2}^{u(k(w_{m-1}))} n_j^{-1} \right) < n_{j_i}^{\gamma/3} n_{j_i}^{-2\gamma/3} = n_{j_i}^{-\gamma/3}.$$

Lastly, suppose $\ell \in (w_{m-1}, w_m) \cap S$ for some $m \geq 1$ and $w_m < j_{i+1}$. Then $k(\ell) = k(w_{m-1})$, and by (5.6) we have

$$\pi_\ell \pi_{k+1}^{-(1+\gamma)} \leq \pi_{w_m-1} \pi_{k(w_{m-1})+1}^{-(1+\gamma)}$$
$$= \left(\prod_{j=1}^{k(w_{m-1})+1} n_j^{-\gamma} \right) \left(\prod_{j=k(w_{m-1})+2}^{u(k(w_{m-1}))-1} n_j \right) < n_{j_i}^{-\gamma} n_{j_i}^{2\gamma/3} = n_{j_i}^{-\gamma/3},$$

and

$$\pi_k \pi_\ell^{-1} \leq \pi_{k(w_{m-1})} \pi_{w_{m-1}}^{-1} \leq n_{j_i}^{-\gamma/3}.$$

Hence, our procedure of choosing $k(\ell)$ satisfies (5.5) for all $\ell \in (j_i, j_{i+1}]$, and the proof is complete. $\qquad \square$

Proof of Lemma 5.2.

We first note that $\mathbb{E}\left[X_g^\ell\right] = p_\ell$ for all $g \in G_\ell$, so

$$b_1 = \sum_{g \in G_\ell} \sum_{h \in B_g} p_\ell^2 = \pi_\ell \pi_k p_\ell^2 = \pi_k \pi_\ell^{-1} \lambda(\ell)^2$$

as claimed.

We now move on to bound $b_2 = \sum_{g \in G_\ell} \sum_{h \in B_g \setminus \{g\}} p_{gh}$. For $g \in G_\ell$, define

$$Y_g^j = \sum_{i=0}^{j} \alpha_i \varepsilon(g_i) \quad \text{and} \quad T_g^j = \sum_{i=j+1}^{\infty} \alpha_i \varepsilon(g_i), \quad j \geq 0. \tag{5.8}$$

For any $g \in G_\ell$ and $h \in B_g \setminus \{g\}$, there is a minimal $0 \leq j \leq k-1$ such that

$$U_g = Y_g^j + T_g^j \quad \text{and} \quad U_h = Y_h^j + T_g^j.$$

Proposition 6.2 implies that

$$p_{gh} = \mathbb{E}\left[X_g^\ell X_h^\ell\right] = \mathbb{P}\left\{Y_g^j + T_g^j \in I_\ell, Y_h^j + T_g^j \in I_\ell\right\}$$

$$= \mathbb{E}\left[\mathbb{P}\left\{Y_g^j + T_g^j \in I_\ell, Y_h^j + T_g^j \in I_\ell \,\middle|\, T_g^j\right\}\right] = \mathbb{E}\left[\mathbb{P}\left\{Y_g^j \in I_\ell - T_g^j \,\middle|\, T_g^j\right\}^2\right],$$

where the last equality used the fact that Y_g^j and Y_h^j are independent and equal in distribution.

Arguing as in (4.8) shows that Y_g^j has density f_j satisfying $\|f_j\|_\infty \leq \frac{1}{\alpha_0} \|\eta_\varepsilon\|_\infty$. Thus, since Y_g^j and T_g^j are independent, we have $\mathbb{P}\left\{Y_g^j \in I_\ell - T_g^j \,\middle|\, T_g^j\right\} = \varphi\left(T_g^j\right)$ where

$$\varphi(t) = \mathbb{P}\left\{Y_g^j \in I_\ell - t\right\} = \int_{I_\ell - t} f_j(x) dx \leq \|f_j\|_\infty |I_\ell - t| = \frac{1}{\alpha_0} \|\eta_\varepsilon\|_\infty c\pi_\ell^{-1}.$$

It follows that

$$b_2 = \sum_{g \in G_\ell} \sum_{h \in B_g \setminus \{g\}} p_{gh} = \pi_\ell \sum_{j=0}^{k-1} (n_{j+1} - 1)\pi_j \mathbb{E}\left[\mathbb{P}\left\{Y_g^j \in I_\ell - T_g^j \,\middle|\, T_g^j\right\}^2\right]$$

$$\leq \pi_\ell(\pi_k - 1)\frac{1}{\alpha_0^2} \|\eta_\varepsilon\|_\infty^2 c^2 \pi_\ell^{-2} \leq \pi_k \frac{1}{\alpha_0^2} \|\eta_\varepsilon\|_\infty^2 c^2 \pi_\ell^{-1}$$

where the first inequality used $\sum_{j=0}^{k-1} (n_{j+1} - 1)\pi_j = \pi_k - 1$. Observe that this bound is of the correct order of magnitude since Jensen's inequality shows that

$$\mathbb{E}\left[\mathbb{P}\left\{Y_g^j \in I_\ell - T_g^j \,\middle|\, T_g^j\right\}^2\right] \geq \mathbb{E}\left[\mathbb{P}\left\{Y_g^j \in I_\ell - T_g^j \,\middle|\, T_g^j\right\}\right]^2 = p_\ell^2 = O(\pi_\ell^{-2}).$$

We finish with the bound on b_3. Let $g^{(1)}, \ldots, g^{(m)}$ be a transversal for G_k in G_ℓ with $g^{(1)} = e$ and $m = \pi_\ell \pi_k^{-1}$. Then for any $g \in G_\ell$, the dependency neighborhood B_g is one of $B^{(1)}, \ldots, B^{(m)}$, where

$$B^{(i)} := g^{(i)} + G_k, \quad 1 \leq i \leq m.$$

Let $\mathcal{G}^{(i)}$ be any sub-σ-field with respect to which $V^{(i)} := V_{g^{(i)}} = \sum_{g \in G_\ell \backslash B^{(i)}} X_g^\ell$ is measurable. Then, noting that

$$\left(V^{(i)}, \sum_{g \in B^{(i)}} X_g^\ell \right) \stackrel{d}{=} \left(V^{(j)}, \sum_{g \in B^{(j)}} X_g^\ell \right)$$

for all $1 \le i, j \le m$, we have

$$\sum_{g \in G_\ell} \mathbb{E}\left[(X_g^\ell - p_\ell) f(V_g + 1) \right] = \sum_{i=1}^m \mathbb{E}\left[f(V^{(i)} + 1) \sum_{g \in B^{(i)}} \left(X_g^\ell - p_\ell \right) \right]$$

$$= \pi_\ell \pi_k^{-1} \mathbb{E}\left[f(V^{(1)} + 1) \sum_{g \in G_k} \left(X_g^\ell - p_\ell \right) \right]$$

$$= \pi_\ell \pi_k^{-1} \mathbb{E}\left[f(V^{(1)} + 1) \mathbb{E}\left[\sum_{g \in G_k} \left(X_g^\ell - p_\ell \right) \Big| \mathcal{G}^{(1)} \right] \right]$$

$$\le \pi_\ell \pi_k^{-1} \|f\|_\infty \mathbb{E}\left| \mathbb{E}\left[\sum_{g \in G_k} \left(X_g^\ell - p_\ell \right) \Big| \mathcal{G}^{(1)} \right] \right|.$$

Recalling the definition of T_g^k from (5.8), observe that as g ranges over G_k, Y_g^k varies but T_g^k does not. By construction, $V^{(1)}$ is measurable with respect to the σ-field generated by T_g^k and $\{\varepsilon(g_j)\}_{g \in G_\ell \backslash G_k, j \ge 0}$. We set $\mathcal{G}^{(1)}$ equal to this σ-field. Since (X_g, T_g^k) is independent of $\{\varepsilon(g_j)\}_{g \in G_\ell \backslash G_k, j \ge 0}$ for each $g \in G_k$, Proposition 6.1 implies that

$$\mathbb{E}\left[\sum_{g \in G_k} \left(X_g^\ell - p_\ell \right) \Big| \mathcal{G}^{(1)} \right] = \mathbb{E}\left[\sum_{g \in G_k} \left(X_g^\ell - p_\ell \right) \Big| T_g^k \right] = \sum_{g \in G_k} \left(\mathbb{E}\left[X_g^\ell \Big| T_g^k \right] - p_\ell \right).$$

It follows that

$$\pi_\ell \pi_k^{-1} \|f\|_\infty \mathbb{E}\left| \mathbb{E}\left[\sum_{g \in G_k} (X_g^\ell - p_\ell) \Big| \mathcal{G}^{(1)} \right] \right| = \pi_\ell \pi_k^{-1} \|f\|_\infty \mathbb{E}\left| \sum_{g \in G_k} \left(\mathbb{E}\left[X_g^\ell | T_g^k \right] - p_\ell \right) \right|$$

$$\le \pi_\ell \pi_k^{-1} \|f\|_\infty \sum_{g \in G_k} \mathbb{E}\left| \left(\mathbb{E}\left[X_g^\ell | T_g^k \right] - p_\ell \right) \right|$$

$$= \pi_\ell \|f\|_\infty \mathbb{E}\left| \left(\mathbb{E}\left[X_g^\ell | T_g^k \right] - p_\ell \right) \right|, \qquad (5.9)$$

so

$$b_3 \le \pi_\ell \|f\|_\infty \mathbb{E}\left| \mathbb{E}\left[X_g^\ell | T_g^k \right] - p_\ell \right|$$

$$\le \pi_\ell \left(1 \wedge \lambda(\ell)^{-\frac{1}{2}} \right) \mathbb{E}\left| \mathbb{P}\left\{ Y_g^k \in I_\ell - T_g^k | T_g^k \right\} - p_\ell \right|.$$

Let ν_k denote the distribution of T_g^k and let $\widetilde{T}_g^k$ be an independent copy of T_g^k. Observe that $T_g^k \in [-a_k, a_k]$ for all $g \in G_\ell$, where

$$a_k := \sum_{i=k+1}^{\infty} \alpha_i.$$

We see that

$$\mathbb{E} \left| \mathbb{P} \left\{ Y_g^k \in I_\ell - T_g^k | T_g^k \right\} - p_\ell \right|$$

$$= \int_{-a_k}^{a_k} \left| \mathbb{P} \left\{ Y_g^k \in I_\ell - t \right\} - \int_{-a_k}^{a_k} \mathbb{E} \left[X_g^\ell | \widetilde{T}_g^k = s \right] d\nu_k(s) \right| d\nu_k(t)$$

$$\leq \int_{-a_k}^{a_k} \int_{-a_k}^{a_k} \left| \mathbb{P} \left\{ Y_g^k \in I_\ell - t \right\} - \mathbb{P} \left\{ Y_g^k \in I_\ell - s \right\} \right| d\nu_k(s) d\nu_k(t)$$

$$= \mathbb{E}_{T_g^k, \widetilde{T}_g^k} \left| \mathbb{P} \left\{ Y_g^k \in \{I_\ell - T_g^k\} \setminus \{I_\ell - \widetilde{T}_g^k\} \right\} - \mathbb{P} \left\{ Y_g^k \in \{I_\ell - \widetilde{T}_g^k\} \setminus \{I_\ell - T_g^k\} \right\} \right|.$$

Because $T_g^k \in [-a_k, a_k]$, we have

$$\{I_\ell - T_g^k\} \supseteq \left[t_0 - \frac{c}{2\pi_\ell} + a_k, t_0 + \frac{c}{2\pi_\ell} - a_k \right],$$

$$\{I_\ell - T_g^k\} \subseteq \left[t_0 - \frac{c}{2\pi_\ell} - a_k, t_0 + \frac{c}{2\pi_\ell} + a_k \right],$$

hence

$$\{I_\ell - T_g^k\} \setminus \{I_\ell - \widetilde{T}_g^k\}$$

$$\subseteq \left[t_0 - \frac{c}{2\pi_\ell} - a_k, t_0 + \frac{c}{2\pi_\ell} + a_k \right] \setminus \left[t_0 - \frac{c}{2\pi_\ell} + a_k, t_0 + \frac{c}{2\pi_\ell} - a_k \right]$$

$$= \left[t_0 - \frac{c}{2\pi_\ell} - a_k, t_0 - \frac{c}{2\pi_\ell} + a_k \right] \bigcup \left[t_0 + \frac{c}{2\pi_\ell} - a_k, t_0 + \frac{c}{2\pi_\ell} + a_k \right].$$

Therefore,

$$\mathbb{E}_{T_g^k, \widetilde{T}_g^k} \left| \mathbb{P} \left\{ Y_g^k \in \{I_\ell - T_g^k\} \setminus \{I_\ell - \widetilde{T}_g^k\} \right\} - \mathbb{P} \left\{ Y_g^k \in \{I_\ell - \widetilde{T}_g^k\} \setminus \{I_\ell - T_g^k\} \right\} \right|$$

$$\leq 2\mathbb{P} \left\{ Y_g^k \in \left[t_0 - \frac{c}{2\pi_\ell} - a_k, t_0 - \frac{c}{2\pi_\ell} + a_k \right] \bigcup \left[t_0 + \frac{c}{2\pi_\ell} - a_k, t_0 + \frac{c}{2\pi_\ell} + a_k \right] \right\}$$

$$\leq 2\mathbb{P} \left\{ U_g \in \left[t_0 - \frac{c}{2\pi_\ell} - 2a_k, t_0 - \frac{c}{2\pi_\ell} + 2a_k \right] \bigcup \left[t_0 + \frac{c}{2\pi_\ell} - 2a_k, t_0 + \frac{c}{2\pi_\ell} + 2a_k \right] \right\}$$

$$= 8a_k \left(\frac{1}{4a_k} \int_{t_0 - \frac{c}{2\pi_\ell} - 2a_k}^{t_0 - \frac{c}{2\pi_\ell} + 2a_k} \eta(x) dx + \frac{1}{4a_k} \int_{t_0 + \frac{c}{2\pi_\ell} - 2a_k}^{t_0 + \frac{c}{2\pi_\ell} + 2a_k} \eta(x) dx \right)$$

$$\leq 16 \left\| \eta \right\|_\infty K \pi_{k+1}^{-(1+\gamma)}$$

$$\leq 16 \frac{1}{\alpha_0} \left\| \eta_\varepsilon \right\|_\infty K \pi_{k+1}^{-(1+\gamma)},$$

where we recall that $\eta(x)$ is the density of U_g and is bounded by $\frac{1}{\alpha_0} \|\eta_\varepsilon\|_\infty$. This concludes the proof of the lemma. $\qquad\square$

The selection of the dependency neighborhoods in Lemma 5.3 is based on the upper bounds derived in the above proof. Although the bounds on b_1 and b_2 are of the correct order of magnitude, it is possible that our use of the triangle inequality in (5.9) may have considerably affected the quality of the bound on b_3. Ideally, we would like to keep the summation inside the absolute value in order to capitalize on the resulting cancellation, but it is not clear how to estimate the left hand side of (5.9) directly. In the original statement of Theorem 5.1, the triangle inequality is also employed [4], but the choice of dependency neighborhoods in our case potentially obviates this step. This observation may be useful in other Poisson approximation problems in which there is long range dependence and the dependency neighborhoods partition the collection of indicators.

6. Appendix

The following propositions are easy results about conditional expectation which are needed in the preceding analysis. In both cases, we assume that the relevant random variables are real-valued and defined on an underlying probability space denoted $(\Omega, \mathcal{F}, P)$.

Proposition 6.1. *If $X \in L^1$ and (X, Y) is independent of $(Z_1, \ldots, Z_m)$, then*
$$\mathbb{E}\left[X \,|\, Y, Z_1, \ldots, Z_m\right] = \mathbb{E}\left[X \,|\, Y\right].$$

Proof. We first observe that $\mathbb{E}\left[X \,|\, Y\right] \in \sigma(Y) \subseteq \sigma(Y, Z_1, \ldots, Z_m)$. Now let $\mathcal{P} = \{A \cap B : A \in \sigma(Y), B \in \sigma(Z_1, \ldots, Z_m)\}$. Then $\mathcal{P}$ is a π-system which generates $\sigma(Y, Z_1, \ldots, Z_m)$. The independence assumption ensures that for any $A \cap B \in \mathcal{P}$,
$$\int_{A \cap B} \mathbb{E}\left[X \,|\, Y\right] dP = \int_A 1_B \mathbb{E}\left[X \,|\, Y\right] dP = P(B) \int_A \mathbb{E}\left[X \,|\, Y\right] dP$$
$$= P(B) \int_A X dP = \int_A 1_B X dP = \int_{A \cap B} X dP.$$

Since the collection of events S for which $\int_S \mathbb{E}\left[X \,|\, Y\right] dP = \int_S X dP$ is a λ-system containing $\mathcal{P}$ (by linearity and dominated convergence), the result follows from Dynkin's theorem. $\qquad\square$

Proposition 6.2. *If X, Y, Z are independent random variables with $X \stackrel{d}{=} Y$, then*
$$\mathbb{E}\left[f\left(X + Z\right) f\left(Y + Z\right)\right] = \mathbb{E}\left[\mathbb{E}\left[f\left(X + Z\right)|\, Z\right]^2\right]$$

for any integrable or nonnegative function f.

Proof. Let μ denote the common distribution of X and Y, and let ν denote the distribution of Z. By independence, (X, Y, Z) has distribution $\mu \times \mu \times \nu$.

Now for every $F \in \sigma(Z)$, there is a Borel set A such that $F = \{Z \in A\}$, so the change of variables formula and Fubini-Tonelli show that
$$\int_F f\left(X + Z\right) f\left(Y + Z\right) dP = \int_A \int_\mathbb{R} \int_\mathbb{R} f(x + z) f(y + z) d\mu(x) d\mu(y) d\nu(z)$$

$$= \int_A \left(\int_{\mathbb{R}} f(x+z) d\mu(x) \right) \left(\int_{\mathbb{R}} f(y+z) d\mu(y) \right) d\nu(z)$$

$$= \int_A \left(\int_{\mathbb{R}} f(x+z) d\mu(x) \right)^2 d\nu(z)$$

$$= \int_A \mathbb{E} \left[f(X+z) \right]^2 d\nu(z) = \int_F g(Z)^2 dP$$

with $g(z) = \mathbb{E} \left[f(X+z) \right]$.

On the other hand, $g(Z) = \mathbb{E} \left[f(X+Z) | Z \right]$ since

$$\int_F f(X+Z) dP = \int_A \int_{\mathbb{R}} f(x+z) d\mu(x) d\nu(z)$$

$$= \int_A \mathbb{E} \left[f(X+z) \right] d\nu(z) = \int_F g(Z) dP.$$

Therefore, $\mathbb{E} \left[f(X+Z) f(Y+Z) | Z \right] = \mathbb{E} \left[f(X+Z) | Z \right]^2$ and the result follows upon taking expectations. $\square$

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